United Kingdom
Mathematics Trust

Senior Mathematical Challenge

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Solutions and investigations

October 4th, 2022

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	C	C	D	C	E	C	E	B	D	A	E	C	A	B	B	B	B	E	D	D	B	A	D	A

1. When the expression $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}$ is simplified, which of the following is obtained?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

Solution **D**

Using the standard factorization $n^2 - 1 = (n-1)(n+1)$, we have

$$\begin{aligned}\frac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)} &= \frac{(1\times3)\times(2\times4)\times(3\times5)\times(4\times6)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}\\ &= \frac{1\times2\times3\times4}{2\times3\times4\times5}\\ &= \frac{1}{5}.\end{aligned}$$

For investigation

1.1 Write the following expressions in simplified form.

(a) $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)\times(6^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)\times(6\times7)}$,

(b) $\dfrac{(2^2-1^2)\times(3^2-2^2)\times(4^2-3^2)\times(5^2-4^2)}{1\times3\times5\times7}$.

(c) $\dfrac{(3^4-1)\times(4^4-1)\times(5^4-1)\times(6^4-1)}{(2\times4)\times(3\times5)\times(4\times6)\times(5\times7)}$.

2. What is the smallest prime which is the sum of five different primes?

A 39 B 41 C 43 D 47 E 53

Solution **C**

If five different primes include 2, they consist of 2 and four odd primes. Hence their sum is even and hence is not prime.

Therefore five different primes whose sum is also prime cannot include 2. Hence they are five odd primes.

The sum of the first five odd primes is $3 + 5 + 7 + 11 + 13 = 39$, which not prime.

The smallest prime greater than 13 is 17. If we replace 13 by 17 we obtain the sum $3 + 5 + 7 + 11 + 17 = 43$, which is prime.

It follows that the smallest prime that is the sum of five different primes is 43.

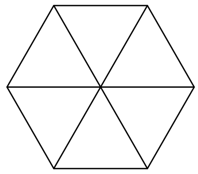
For investigation

2.1 Which is the smallest prime which is the sum of six different primes?

3. The figure shows a regular hexagon.

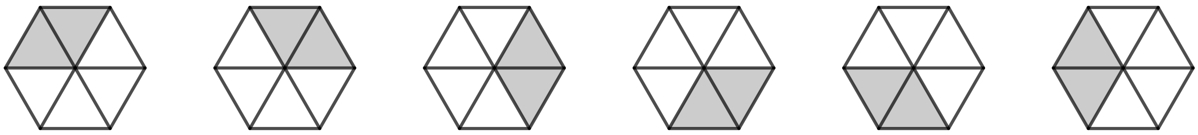
How many parallelograms are there in the figure?

A 2 B 4 C 6 D 8 E more than 8



Solution **C**

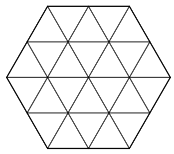
The parallelograms in the figure are made up from pairs of adjacent triangles. Therefore there are six parallelograms in the figure, as shown in the diagrams below.



For investigation

3.1 The figure on the right shows a regular hexagon divided into 24 congruent equilateral triangles.

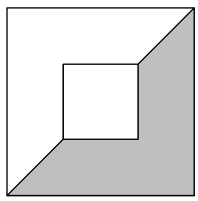
How many parallelograms are there in the figure?



4. The diagram shows two symmetrically placed squares with sides of length 2 and 5.

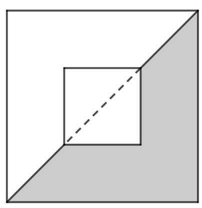
What is the ratio of the area of the small square to that of the shaded region?

A 7 : 24 B 1 : 3 C 8 : 25 D 8 : 21 E 2 : 5



Solution **D**

The small square has area 2^2, that is, 4. The shaded region is made up of half the large square with side length 5, with half the small square with side length 2 removed.



Therefore, the shaded area is $\frac{1}{2}(5^2) - \frac{1}{2}(2^2) = \frac{25}{2} - 2 = \frac{21}{2}$. Hence the ratio of the area of the small square to that of the shaded region is $4 : \frac{21}{2}$, that is, $8 : 21$.

For investigation

4.1 Suppose that the area of the shaded region were one-third of the area of the outer square. What would be the ratio of the area of the inner square to the area of the outer square?

4.2 Suppose that the ratio of the shaded area to the area of the small square is $7 : 18$. What is the ratio of the side length of the small square to the side length of the large square?

5. What is the value of $\dfrac{1}{1.01}+\dfrac{1}{1.1}+\dfrac{1}{1}+\dfrac{1}{11}+\dfrac{1}{101}$?

A 2.9 B 2.99 C 3 D 3.01 E 3.1

SOLUTION **C**

Note:
The key to an efficient solution is to regroup the terms so as to simplify the arithmetic.

We have

$$\begin{aligned}\frac{1}{1.01}+\frac{1}{1.1}+\frac{1}{1}+\frac{1}{11}+\frac{1}{101} &= \left(\frac{1}{1.01}+\frac{1}{101}\right)+\left(\frac{1}{1.1}+\frac{1}{11}\right)+\frac{1}{1}\\ &= \left(\frac{100}{101}+\frac{1}{101}\right)+\left(\frac{10}{11}+\frac{1}{11}\right)+\frac{1}{1}\\ &= \frac{101}{101}+\frac{11}{11}+\frac{1}{1}\\ &= 1+1+1\\ &= 3.\end{aligned}$$

FOR INVESTIGATION

5.1 What is the value of $\dfrac{1}{1.24}+\dfrac{1}{6.2}+\dfrac{1}{31}$?

6. What is the value of $\dfrac{4^{800}}{8^{400}}$?

A $\frac{1}{2^{400}}$ B $\frac{1}{2^{200}}$ C 1 D 2^{200} E 2^{400}

SOLUTION **E**

We have

$$\begin{aligned}\frac{4^{800}}{8^{400}} &= \frac{(2^2)^{800}}{(2^3)^{400}}\\ &= \frac{2^{1600}}{2^{1200}}\\ &= 2^{1600-1200}\\ &= 2^{400}.\end{aligned}$$

FOR INVESTIGATION

6.1 For which integer n is $\dfrac{27^{900}}{9^{2700}}=3^n$?

7. In 2021, a first class postage stamp cost 85p and a second class postage stamp cost 66p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

A 10 B 8 C 7 D 5 E 2

SOLUTION **C**

The cost of r first class postage stamps at 85p each, and s second class postage stamps at 66p each is $(85r + 66s)$p.

This is an exact number of pounds provided that

$$85r + 66s = 100t \qquad (1)$$

for some positive integer t.

We therefore seek the solution of (1) in which r, s and t are positive integers with $r + s$ as small as possible.

Equation (1) may be rearranged as

$$85r = 100t - 66s. \qquad (2)$$

Since 100 and 66 are both divisible by 2, it follows from (2) that $85r$ is divisible by 2. Therefore r is divisible by 2. Hence $r \geq 2$.

Equation (1) may also be rearranged as

$$66s = 100t - 85r. \qquad (3)$$

Since 100 and 85 are both divisible by 5, it follows from (3) that $66s$ is divisible by 5. Therefore s is divisible by 5. Hence $s \geq 5$.

We now note that when $r = 2$ and $s = 5$, we have

$$85r + 66s = 85 \times 2 + 66 \times 5 = 170 + 330 = 500.$$

Therefore equation (1) has the solution $r = 2$, $s = 5$ and $t = 5$. Because $r = 2$ and $s = 5$ are the least possible values for r and s, it follows that $r + s$ has the least possible value among the solutions of (1) in which r, s and t are positive integers.

Since, in this case, $r + s = 2 + 5 = 7$, we deduce that the least number of stamps that should be purchased is 7.

FOR INVESTIGATION

7.1 Today a first class postage stamp costs 95p, and a second class postage stamp costs 68p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

7.2 Find the solution of the equation

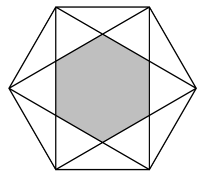
$$45r + 56s = 100t$$

in which r, s and t are positive integers and $r + s$ is as small as possible.

8. In the diagram, the outer hexagon is regular and has an area of 216.

What is the shaded area?

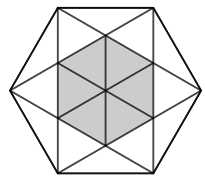
A 108 B 96 C 90 D 84 E 72



Solution **E**

Method 1

In the diagram on the right we have added three lines to the diagram given in the question.



The outer hexagon is now divided into 12 congruent equilateral triangles and 6 congruent triangles with angles 120°, 30° and 30°.

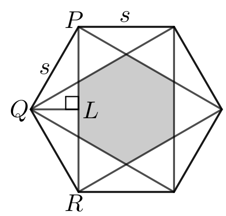
We leave it to the reader to check that each of the triangles with angles 120°, 30°, and 30° has the same area as each of the equilateral triangles (see Problem 8.1).

It follows that the outer hexagon is divided into 18 triangles all with the same area.

The shaded area is made up of 6 of these triangles. Hence its area is $\frac{6}{18}$, that is, $\frac{1}{3}$ of the area of the outer hexagon. Therefore the area of the shaded hexagon is $\frac{1}{3} \times 216 = 72$.

Method 2

We let s be the side length of the outer hexagon, and P, Q and R be three adjacent vertices of this hexagon, as shown.



We let L be the midpoint of PR.

We leave it to the reader to check that PLQ is a right-angled triangle in which $\angle PQL = 60°$ and $PL = \frac{\sqrt{3}}{2}s$ (see Problem 8.2). It follows that $PR = \sqrt{3}s$.

The inner hexagon is regular. The distance between its parallel sides is s. The outer hexagon is regular. The distance between its parallel sides is $\sqrt{3}s$. The ratio of the areas of similar figures is the same as the ratio of the squares of their corresponding lengths. Therefore

$$\text{area of inner hexagon} : \text{area of outer hexagon} = s^2 : (\sqrt{3}s)^2 = s^2 : 3s^2 = 1 : 3.$$

It follows that the area of the shaded hexagon is $\frac{1}{3} \times 216 = 72$.

For investigation

8.1 Show that in the diagram of Method 1, all the 18 triangles into which the outer hexagon is divided have the same area.

8.2 Show that in the diagram of Method 2, the triangle PLQ is right-angled, $\angle PQL = 60°$ and $PL = \frac{\sqrt{3}}{2}s$

9. A light-nanosecond is the distance that a photon can travel at the speed of light in one billionth of a second. The speed of light is $3 \times 10^8\,\mathrm{ms}^{-1}$.

How far is a light-nanosecond?

A 3 cm B 30 cm C 3 m D 30 m E 300 m

Solution **B**

One billionth of a second is $\dfrac{1}{10^9}$ seconds.

Hence, in one billionth of a second light travels

$$\frac{1}{10^9} \times (3 \times 10^8)\,\mathrm{m} = \frac{3}{10}\,\mathrm{m} = 30\,\mathrm{cm}.$$

Therefore a light-nanosecond is 30 cm.

For investigation

9.1 A light-minute is the distance that a photon travels in one minute at the speed of light. The mean distance of the Earth from the Sun is approximately 150 million kilometres.

How many light-minutes is that?

10. What is the value of x in the equation $\dfrac{1 + 2x + 3x^2}{3 + 2x + x^2} = 3$?

A -5 B -4 C -3 D -2 E -1

Solution **D**

We note first that $3 + 2x + x^2 = 2 + (1 + x)^2$. Therefore for all real numbers x, we have $3 + 2x + x^2 \geq 2$. Hence for all real numbers x, it follows that $3 + 2x + x^2 \neq 0$.

Therefore we can multiply both sides of the equation given in the question by $3 + 2x + x^2$. In this way, we have

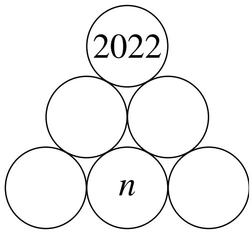
$$\begin{aligned}\frac{1 + 2x + 3x^2}{3 + 2x + x^2} = 3 &\Leftrightarrow 1 + 2x + 3x^2 = 3(3 + 2x + x^2)\\ &\Leftrightarrow 1 + 2x + 3x^2 = 9 + 6x + 3x^2\\ &\Leftrightarrow 4x + 8 = 0\\ &\Leftrightarrow x = -2.\end{aligned}$$

For investigation

10.1 Find all the solutions of the following equations.

(a) $\dfrac{1 + 3x + 5x^2}{5 + 3x + x^2} = 1.$ (b) $\dfrac{5 + 7x + 2x^2}{7 + 10x + 3x^2} = 1.$

11. In the number triangle shown, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.

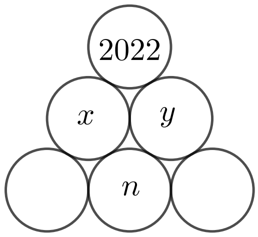


What is the value of n?

A 1 B 2 C 3 D 6 E 33

Solution **A**

We let x and y be the positive integers in the discs in the middle row, as shown.



Since these are the products of the two numbers immediately below, n is a factor of both x and y. Hence n^2 is a factor of xy.

Now $xy = 2022$. It follows that n^2 is a factor of 2022.

The prime factorization of 2022 is

$$2022 = 2 \times 3 \times 337.$$

It follows that 1^2 is the only square of a positive integer that is a factor of 2022.

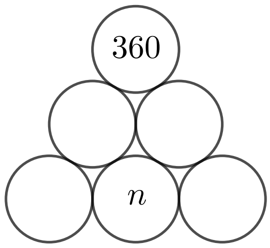
We deduce that $n = 1$.

For investigation

11.1 Check that $2 \times 3 \times 337$ is the factorization of 2022 into prime factors.

11.2 In the number triangle shown on the right, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.

What are the possible values of n?



12. What is the sum of the digits of the integer which is equal to $6666666^2 - 3333333^2$?

A 27 B 36 C 45 D 54 E 63

SOLUTION **E**

Using the standard factorization $x^2 - y^2 = (x - y)(x + y)$, we have

$$\begin{aligned} 6666666^2 - 3333333^2 &= (6666666 - 3333333)(6666666 + 3333333) \\ &= 3333333 \times 9999999 \\ &= 3333333 \times (10000000 - 1) \\ &= 3333333 \times 10000000 - 3333333 \times 1 \\ &= 33333330000000 - 3333333 \\ &= 33333326666667. \end{aligned}$$

It may be seen that the number 33333326666667 is written with six 3s followed by one 2, six 6s and one 7. Therefore the sum of its digits is

$$\begin{aligned} (6 \times 3) + 2 + (6 \times 6) + 7 &= 18 + 2 + 36 + 7 \\ &= 63. \end{aligned}$$

FOR INVESTIGATION

12.1 Let $n = 666\,666\,666^2 - 333\,333\,333^2$.

What is the sum of the digits of n?

12.2 (a) Let a be the integer $\overbrace{666\ldots666}^{k}$ which in standard form is written as a string of k 6s, and let b be the integer $\overbrace{333\ldots333}^{k}$ which is written as a string of k 3s.

Find a formula, in terms of k, for the sum of the digits of the integer $a^2 - b^2$.

(b) Find a formula for the sum of the digits of $c^2 - d^2$ where $c = \overbrace{777\ldots777}^{k}$ and $d = \overbrace{222\ldots222}^{k}$.

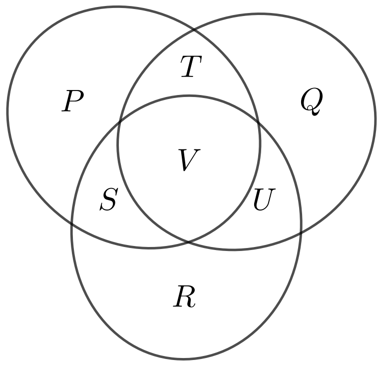
13. Three rugs have a combined area of $90\,\text{m}^2$. When they are laid down to cover completely a floor of area $60\,\text{m}^2$, the area which is covered by exactly two layers of rug is $12\,\text{m}^2$.

What is the area of floor covered by exactly three layers of rug?

A $2\,\text{m}^2$ B $6\,\text{m}^2$ C $9\,\text{m}^2$ D $10\,\text{m}^2$ E $12\,\text{m}^2$

SOLUTION **C**

The diagram shows the three overlapping rugs.



We let the areas of the different regions, in m^2, be P, Q, R, S, T, U and V, as shown.

The three rugs have areas $P + S + T + V$, $Q + T + U + V$ and $R + S + U + V$. Therefore, because the three rugs have a combined area of $90\,\text{m}^2$,

$$(P + S + T + V) + (Q + T + U + V) + (R + S + U + V) = 90.$$

This equation may be rearranged to give

$$(P + Q + R) + 2(S + T + U) + 3V = 90. \qquad (1)$$

Because the floor has an area of $60\,\text{m}^2$,

$$(P + Q + R) + (S + T + U) + V = 60. \qquad (2)$$

By subtracting equation (2) from equation (1), we obtain

$$(S + T + U) + 2V = 30. \qquad (3)$$

Because the area covered by exactly two layers of rug is $12\,\text{m}^2$,

$$S + T + U = 12. \qquad (4)$$

By subtracting equation (4) from equation (3), we have $2V = 18$. Therefore $V = 9$.

Therefore the area of floor covered by exactly three layers of rug is $9\,\text{m}^2$.

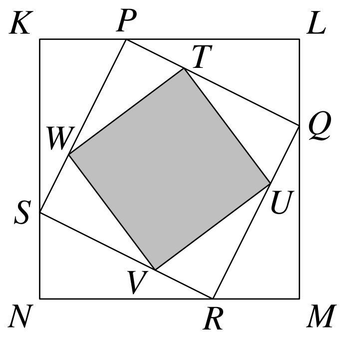
FOR INVESTIGATION

13.1 What is the area of the floor that is covered by exactly one rug?

14. The diagram shows a square, $KLMN$. A second square $PQRS$ is drawn inside it, as shown in the diagram, where P divides the side KL in the ratio $1 : 2$. Similarly, a third square $TUVW$ is drawn inside $PQRS$ with T dividing PQ in the ratio $1 : 2$.

What fraction of the area of $KLMN$ is shaded?

A $\frac{25}{81}$ B $\frac{16}{49}$ C $\frac{4}{9}$ D $\frac{40}{81}$ E $\frac{2}{3}$



Solution **A**

Since P divides the side KL in the ratio $1 : 2$, we choose units so that KL has length 3. Hence KP has length 1, and PL has length 2. We note also that it follows that the square $KLMN$ has area 3^2, that is, 9.

We leave it to the reader to show that the triangles PKS, QLP, RMQ and SNR are congruent.

It follows, in particular, that $SK = PL$. Therefore SK has length 2.

Using Pythagoras' Theorem, applied to the right-angled triangle PKS, we have $PS^2 = 1^2 + 2^2 = 5$. Therefore the square $PQRS$ has area 5.

Therefore the area of the square $PQRS$ as a fraction of the area of the square $KLMN$ is $\frac{5}{9}$.

It follows, similarly, that the area of the square $TUVW$ as a fraction of the area of the square $PQRS$ is also $\frac{5}{9}$.

It follows that the area of the shaded square $TUVW$ is $\frac{5}{9} \times \frac{5}{9} \times$ the area of the square $KLMN$.

Since $\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$, it follows that the fraction of the area of $KLMN$ that is shaded is $\frac{25}{81}$.

For investigation

14.1 Show that the triangles PKS, QLP, RMQ and SNR are congruent.

14.2 Suppose that P divides the side KL in the ratio $1 : 3$, and that T divides PQ in the ratio $1 : 3$.

In this case, what fraction of the area of $KLMN$ is shaded?

14.3 Suppose that P divides the side KL in the ratio $p : q$, and that T divides PQ in the ratio $r : s$.

In this case, what fraction of the area of $KLMN$ is shaded?

15. The hare and the tortoise had a race over 100 m, in which both maintained constant speeds. When the hare reached the finish line, it was 75 m in front of the tortoise. The hare immediately turned around and ran back towards the start line.

How far from the finish line did the hare and the tortoise meet?

A 54 B 60 C 64 D $66\frac{2}{3}$ E 72

SOLUTION **B**

The tortoise is 75 m behind the hare when the hare has run the full 100 m. Therefore the tortoise runs 25 m in the same time as the hare runs 100 m. Hence the hare runs at four times the speed of the tortoise.

When the hare turns round, the hare and the tortoise are 75 m apart and running towards each other.

Since the hare is running at four times the speed of the tortoise, when they meet the hare has run $\frac{4}{5}$ths of the 75 m they were apart when the hare turned around. The tortoise has run $\frac{1}{5}$th of this distance.

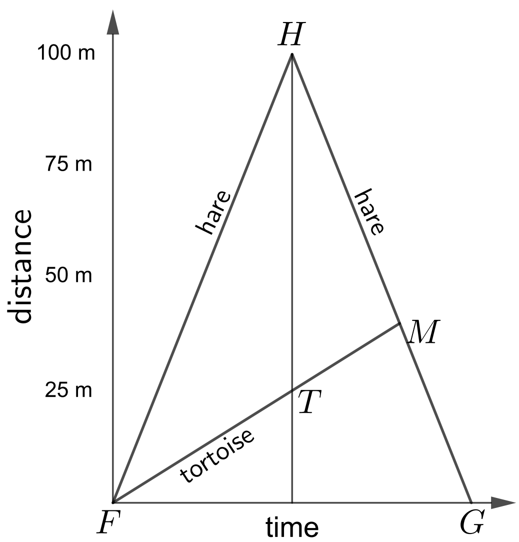
Therefore they meet at a distance $\frac{4}{5} \times 75$ m, that is, 60 m, from the finish line.

FOR INVESTIGATION

15.1 The distance-time graph below illustrates the paths of the hare and the tortoise.

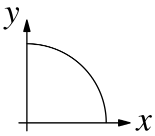
The point H corresponds to the position of the hare when the hare reaches the finish line and turns around. The point T corresponds to the position of the tortoise at this time. The point M corresponds to their common position when they meet. The points F and G are as shown.

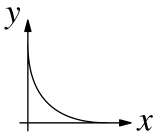
Use the geometry of the diagram to work out how far from the finish line are the hare and tortoise when they meet.

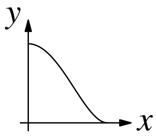


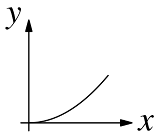
[*Hint*: Extend the line FM to meet the line through G which is perpendicular to FG.]

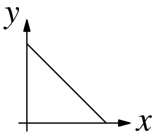
16. Which diagram could be a sketch of the curve $\sqrt{x} + \sqrt{y} = 1$?

A 

B 

C 

D 

E 

Solution **B**

Commentary

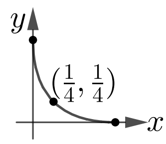
In a question of this type you are not expected to sketch the curve. Instead, you need to look for some feature of the equation that enables you quickly to rule out all but one of the diagrams given as options.

In this case it may be seen that the curve corresponding to the equation $\sqrt{x} + \sqrt{y} = 1$ has reflectional symmetry in the line $y = x$, and that the points $(1, 0)$ and $(0, 1)$ lie on the curve.

Unfortunately these facts are compatible with any of the options A, B and E being the sketch of the curve. So we need to find another fact to eliminate all but one of these options.

It turns out that one way to do this is to find a point on the curve different from $(1, 0)$ and $(0, 1)$.

Note that $\sqrt{\frac{1}{4}} = \frac{1}{2}$. Hence $\sqrt{\frac{1}{4}} + \sqrt{\frac{1}{4}} = 1$.



It follows that the point $(\frac{1}{4}, \frac{1}{4})$ lies on the curve with equation $\sqrt{x} + \sqrt{y} = 1$.

It may been seen that this rules out all the graphs given as options, other than the graph given in option B.

For investigation

16.1 For each of the following equations determine which, if any, of the diagrams given in this question might be a sketch of all, or part, of the curve corresponding to the equation.

(a) $y = x^3$,

(b) $y = \frac{1}{2}x^2$,

(c) $x^2 + y^2 = 1$,

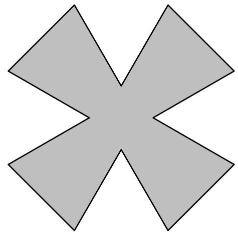
(d) $x + y = 1$,

(e) $x^2 - y^2 = 1$,

(f) $y = \frac{1}{2}(1 + \cos x)$.

17. The shape shown is made by removing four equilateral triangles with side-length 1 from a regular octagon with side-length 1.

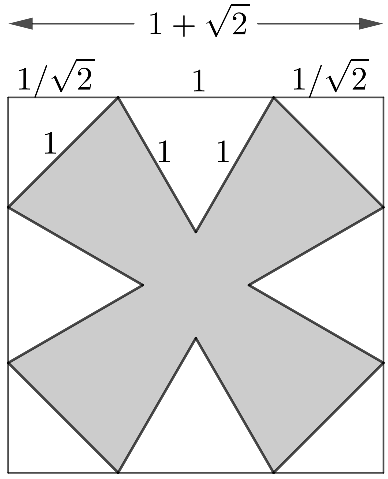
What is the area of the shape?



A $2 - 2\sqrt{2} + \sqrt{3}$ B $2 + 2\sqrt{2} - \sqrt{3}$ C $2 + 2\sqrt{2} + \sqrt{3}$
D $3 - 2\sqrt{2} - \sqrt{3}$ E $2 - 2\sqrt{2} - \sqrt{3}$

Solution **B**

As the diagram on the right shows, a regular octagon with side length 1 may be obtained by cutting four triangular corners from the square.



Each of these corners is an isosceles right-angled triangles with a hypotenuse of length 1. It follows, by Pythagoras' Theorem that the other two sides of these triangles have length $1/\sqrt{2}$.

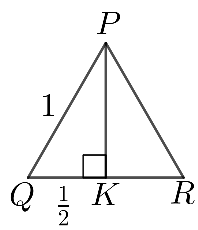
It follows that the side length of the square is given by $1/\sqrt{2} + 1 + 1/\sqrt{2} = 1 + \sqrt{2}$. Therefore the area of the square is $(1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$.

The four triangular corners fit together to make a square of side length 1 and hence area 1.

Therefore the area of the octagon is $(3 + 2\sqrt{2}) - 1 = 2 + 2\sqrt{2}$.

From this area we need to subtract the area of the four equilateral triangles that are removed from the octagon to make the shaded shape.

We let PQR be an equilateral triangle with side length 1, and let PK be the perpendicular from P to QR as shown.



It can be checked that K is the midpoint of QR (see Problem 17.1) and hence that QK has length $\frac{1}{2}$. By Pythagoras' Theorem applied to the right-angled triangle PQK we have $QK^2 + PK^2 = PQ^2$. Therefore $PK^2 = PQ^2 - QK^2 = 1 - \frac{1}{2}^2 = \frac{3}{4}$.

Therefore $PK = \frac{1}{2}\sqrt{3}$.

We can now deduce that the area of the triangle PQR is $\frac{1}{2}(QR \times PK) = \frac{1}{2}(1 \times \frac{1}{2}\sqrt{3}) = \frac{1}{4}\sqrt{3}$.

It now follows that the area of the shape is

$$2 + 2\sqrt{2} - 4 \times \tfrac{1}{4}\sqrt{3} = 2 + 2\sqrt{2} - \sqrt{3}.$$

For investigation

17.1 Prove that the triangles PKQ and PKR are congruent. Deduce that the length of QK is $\frac{1}{2}$.

17.2 Use the $\frac{1}{2}ab\sin C$ formula for the area of a triangle to confirm that the area of an equilateral triangle with side length 1 is $\frac{1}{4}\sqrt{3}$.

18. The numbers x and y are such that $3^x + 3^{y+1} = 5\sqrt{3}$ and $3^{x+1} + 3^y = 3\sqrt{3}$.

What is the value of $3^x + 3^y$?

A $\sqrt{3}$ B $2\sqrt{3}$ C $3\sqrt{3}$ D $4\sqrt{3}$ E $5\sqrt{3}$

Solution **B**

Note that $3^{x+1} = 3(3^x)$ and $3^{y+1} = 3(3^y)$.

Therefore the equations given in the question may be rewritten as

$$3^x + 3(3^y) = 5\sqrt{3} \qquad (1)$$

and

$$3(3^x) + 3^y = 3\sqrt{3}. \qquad (2)$$

Adding equations (1) and (2) gives

$$4(3^x + 3^y) = 8\sqrt{3}. \qquad (3)$$

Therefore

$$3^x + 3^y = 2\sqrt{3}. \qquad (4)$$

For investigation

18.1 (a) Use equations (1) and (2) to find the values of 3^x and 3^y.

(b) Check that the values for 3^x and 3^y that you have found satisfy equation (4).

18.2 (For those who know about logarithms.)

(a) Use your answer to Problem 18.1 (a) to show that $x = \frac{1}{2} - \frac{\ln 2}{\ln 3}$.

(b) Find a similar expression for the value of y.

18.3 Find the values of x and y that satisfy both of the equations

$$4^{x+1} + 5^y = 281$$

and

$$4^x + 5^{y+1} = 189.$$

19. How many pairs of real numbers (x, y) satisfy the simultaneous equations $x^2 - y = 2022$ and $y^2 - x = 2022$?

A infinitely many B 1 C 2 D 3 E 4

Solution **E**

By subtracting the equation $y^2 - x = 2022$ from the equation $x^2 - y = 2022$, we obtain

$$x^2 - y^2 + x - y = 0. \qquad (1)$$

The left hand side of equation (1) factorizes to give

$$(x - y)(x + y + 1) = 0. \qquad (2)$$

It follows that either $x - y = 0$ or $(x + y + 1) = 0$. That is, either $y = x$ or $y = -x - 1$.

When $y = x$ both equations of the question are equivalent to the equation $x^2 - x = 2022$. We can rewrite this equation as

$$x^2 - x - 2022 = 0. \qquad (3)$$

Equation (3) is a quadratic equation of the form $ax^2 + bx + c = 0$, with $a = 1$, $b = -1$ and $c = -2022$. Therefore $b^2 - 4ac = (-1)^2 + 4 \times 2022$ which is greater than 0.

It follows that equation (3) has two distinct real number solutions, say x_1 and x_2. It follows that the two pairs of real numbers (x_1, x_1) and (x_2, x_2) satisfy the simultaneous equations of the question.

It can be checked that when $y = -x - 1$ both equations of the question are equivalent to the equation

$$x^2 + x - 2021 = 0. \qquad (4)$$

Equation (4) is a quadratic equation of the form $ax^2 + bx + c = 0$, with $a = 1$, $b = 1$ and $c = -2021$. Therefore $b^2 - 4ac = 1^2 + 4 \times 2021$ which is greater than 0.

It follows that equation (4) has two distinct real number solutions, say x_3 and x_4. It follows that the two pairs of real numbers $(x_3, -x_3 - 1)$ and $(x_4, -x_4 - 1)$ satisfy the simultaneous equations of the question.

In these last two solutions $y \neq x$ and therefore they are different from the first two solutions.

We can therefore conclude that there are four pairs of real numbers that satisfy the simultaneous equations of the question.

For investigation

19.1 Check that when $y = -x - 1$ both equations (1) and (2) are equivalent to the equation $x^2 + x - 2021 = 0$.

19.2 Consider the curves corresponding to the equations $x^2 - y = 2022$ and $y^2 - x = 2022$.

(a) What is the geometrical relationship between these two curves?

(b) Sketch the two curves.

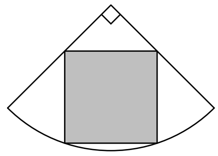
(c) Check that the two curves meet in four distinct points.

19.3 Explain why $b^2 - 4ac > 0$ is the condition for the quadratic equation $ax^2 + bx + c = 0$ to have two distinct real number solutions.

20. A square is inscribed inside a quadrant of a circle. The circle has radius 10.

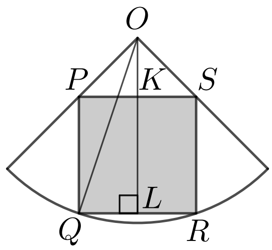
What is the area of the square?

A $25\sqrt{2}$ B 36 C 12π D 40 E $30\sqrt{2}$



SOLUTION **D**

We let O be the centre of the circle, and P, Q, R and S be the vertices of the square, as shown. We let OL be the perpendicular from O to QR, and K be the point where this perpendicular meets PS.



We let s be the side length of the square. It may be checked (see Problem 20.1) that the triangles OLQ and OLR are congruent. It follows that $QL = \frac{1}{2}QR = \frac{1}{2}s = PK$.

It may be checked that OKP is a right-angled isosceles triangle (see Problem 20.2). Therefore $OK = PK = \frac{1}{2}s$.

It follows that in the right-angled triangle OLQ we have $QL = \frac{1}{2}s$, $OL = OK + KL = \frac{1}{2}s + s = \frac{3}{2}s$ and $OQ = 10$.

Therefore, by Pythagoras' Theorem

$$(\tfrac{1}{2}s)^2 + (\tfrac{3}{2}s)^2 = 10^2.$$

Hence

$$\tfrac{1}{4}s^2 + \tfrac{9}{4}s^2 = 100.$$

That is,

$$\tfrac{5}{2}s^2 = 100.$$

Hence

$$s^2 = \tfrac{2}{5} \times 100 = 40.$$

It follows that the area of the square is 40.

FOR INVESTIGATION

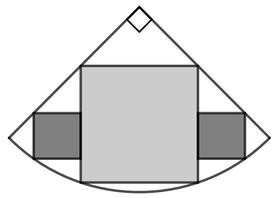
20.1 Show that the triangles OLQ and OLR are congruent.

20.2 Show that OKP is a right-angled isosceles triangle.

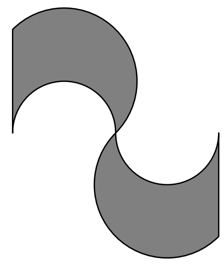
20.3 The diagram on the right shows one larger square and two smaller squares inscribed in the quadrant of a circle.

The circle has radius 10.

Find the side length of the smaller squares.



21. The perimeter of a logo is created from two vertical straight edges, two small semicircles with horizontal diameters and two large semicircles. Both of the straight edges and the diameters of the small semicircles have length 2. The logo has rotational symmetry as shown.

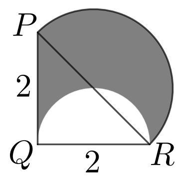


What is the shaded area?

A 4 B $4-\pi$ C 8 D $4+\pi$ E 12

SOLUTION **D**

The diagram on the right shows the top half of the shaded region.



This region is made up of the right-angled triangle PQR and the semicircle with diameter PR, but with the semicircle with QR as diameter removed.

The area of the triangle PQR is $\frac{1}{2}(PQ \times QR) = \frac{1}{2}(2 \times 2) = 2$.

By Pythagoras' Theorem, applied to the right-angled triangle PQR, we have $PR^2 = 2^2 + 2^2 = 8$. Therefore $PR = 2\sqrt{2}$.

Hence the semicircle with diameter PR has radius $\sqrt{2}$. Hence the area of this semicircle is $\frac{1}{2}\left(\pi(\sqrt{2})^2\right) = \pi$.

The semicircle with QR as diameter has radius 1, and therefore its area is $\frac{1}{2}(\pi 1^2) = \frac{1}{2}\pi$.

It follows that the area of the shaded region in the diagram above is

$$2 + \pi - \tfrac{1}{2}\pi = 2 + \tfrac{1}{2}\pi.$$

This area is half of the shaded area in the logo.

Therefore the shaded area in the logo is $2 \times (2 + \frac{1}{2}\pi) = 4 + \pi$.

FOR INVESTIGATION

21.1 Find the length of the perimeter of the shaded area.

22. How many pairs of integers (x, y) satisfy the equation $\sqrt{x - \sqrt{x+23}} = 2\sqrt{2} - y$?

A 0 B 1 C 4 D 8 E infinitely many

SOLUTION **B**

Suppose that (x, y) is a pair of integers that satisfies the equation given in the question. That is, suppose that x and y are integers such that

$$\sqrt{x - \sqrt{x+23}} = 2\sqrt{2} - y. \quad (1)$$

Squaring both sides of equation (1), we obtain

$$x - \sqrt{x+23} = 8 - 4\sqrt{2}y + y^2. \quad (2)$$

We can rewrite equation (2) as

$$\sqrt{x+23} = 4\sqrt{2}y - (8 + y^2 - x). \quad (3)$$

Since x and y are integers, $8 + y^2 - x$ is an integer. For convenience we put $z = 8 + y^2 - x$. Then equation (3) becomes

$$\sqrt{x+23} = 4\sqrt{2}y - z. \quad (4)$$

By squaring both sides of equation (4), we obtain

$$x + 23 = 32y^2 - 8\sqrt{2}yz + z^2. \quad (5)$$

Equation (5) may be rearranged to give

$$8\sqrt{2}yz = 32y^2 + z^2 - x - 23. \quad (6)$$

Now, if $yz \neq 0$, equation (6) implies that

$$\sqrt{2} = \frac{32y^2 + z^2 - x - 23}{8yz}. \quad (7)$$

Because x, y and z are integers, equation (7) implies that $\sqrt{2}$ is rational. Since $\sqrt{2}$ is irrational (see Problem 22.1), we deduce that $yz = 0$. It follows that $y = 0$ or $z = 0$.

Suppose first that $y = 0$. It then follows from equation (2) that

$$x - \sqrt{x+23} = 8. \quad (8)$$

We leave it as an exercise for the reader to show that equation (8) does not have an integer solution (see Problem 22.2).

Now suppose that $z = 0$. Then

$$8 + y^2 - x = 0. \quad (9)$$

It now follows from equations (3) and (9) that

$$\sqrt{x+23} = 4\sqrt{2}y. \quad (10)$$

Squaring both sides of equation (10), we obtain $x + 23 = 32y^2$ and hence

$$x = 32y^2 - 23. \quad (11)$$

We now use equation (11) to substitute $32y^2 - 23$ for x in equation (9). In this way we obtain

$$8 + y^2 - (32y^2 - 23) = 0. \quad (12).$$

Equation (12) may be rearranged to give

$$31(1 - y^2) = 0. \quad (13)$$

Hence, $1 - y^2 = 0$. Therefore, $y = -1$ or $y = 1$. In either case, it follows from the equation (11) that $x = 9$.

It seems that we have found two pairs of integers $(9, -1)$ and $(9, 1)$ that satisfy equation (1). However, we obtained these solutions by squaring both sides of equations (1), (4) and (10).

Squaring both sides of an equation is an operation that runs the risk of introducing spurious solutions. Therefore we need to check whether or not $(9, -1)$ and $(9, 1)$ are solutions of the original equation.

We leave it to the reader (see Problem 22.3) to check that $(9, 1)$ is a solution of equation (1), but $(9, -1)$ is *not* a solution.

It follows that there is just one pair of integers (x, y), namely $(9, 1)$, that satisfies the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$.

FOR INVESTIGATION

22.1 The solution above uses the fact that $\sqrt{2}$ is not a rational number. This means that there do not exist integers p and q such that

$$\sqrt{2} = \frac{p}{q}.$$

Find a proof of this fact. That is, find a proof in a book or on the internet, or ask your teacher.

22.2 Show that the equation

$$x - \sqrt{x + 23} = 8$$

does not have an integer solution.

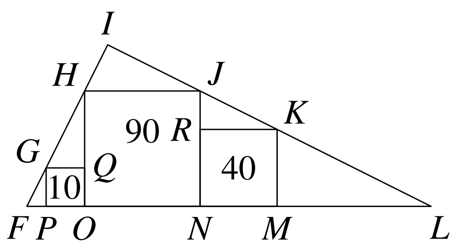
[Hint: rewrite this equation as $x - 8 = \sqrt{x + 23}$. Square both sides of this equation. Then consider the solutions of the quadratic equation that you obtain.]

22.3 Show that $x = 9$, $y = 1$ is a solution of the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$, but $x = 9$, $y = -1$ is not a solution.

23. Three squares $GQOP$, $HJNO$ and $RKMN$ have vertices which sit on the sides of triangle FIL as shown. The squares have areas of 10, 90 and 40 respectively.

What is the area of triangle FIL?

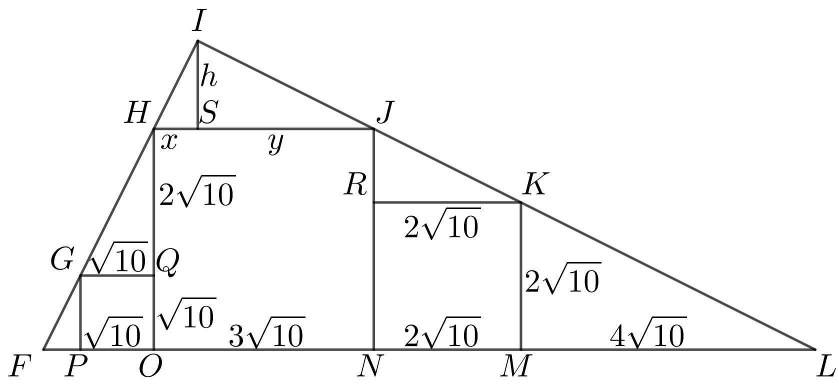
A 220.5 B $\frac{21}{5}\sqrt{10}$ C 252 D $\frac{21}{2}\sqrt{10}$ E 441



Solution **A**

We let S be the foot of the perpendicular from the point I to HJ. We let h, x and y be the lengths of IS, HS and SJ, respectively.

The square $GQOP$ has area 10 and hence side length $\sqrt{10}$.



Similarly, the side length of the square $HJNO$ is $\sqrt{90} = 3\sqrt{10}$ and the side length of the square $RKMN$ is $\sqrt{40} = 2\sqrt{10}$.

We have $GQ = \sqrt{10}$ and $HQ = HO - QO = 3\sqrt{10} - \sqrt{10} = 2\sqrt{10}$. We leave it to the reader to check that the triangles GPF, HQG and ISH are similar. It follows that $FP = \frac{1}{2}\sqrt{10}$ and $h = 2x$.

We also have $JR = JN - RN = 3\sqrt{10} - 2\sqrt{10} = \sqrt{10}$ and $RK = 2\sqrt{10}$. The triangles KML, JRK and ISJ are similar. Therefore $ML = 4\sqrt{10}$ and $h = \frac{1}{2}y$.

Since $h = 2x$ and $h = \frac{1}{2}y$, it follows that $y = 4x$. Also $x + y = 3\sqrt{10}$. Therefore $5x = 3\sqrt{10}$. Hence $x = \frac{3}{5}\sqrt{10}$, $y = \frac{12}{5}\sqrt{10}$ and $h = \frac{6}{5}\sqrt{10}$.

We now have that the base of the triangle FIL is

$$FL = FP + PO + ON + NM + ML = \tfrac{1}{2}\sqrt{10} + \sqrt{10} + 3\sqrt{10} + 2\sqrt{10} + 4\sqrt{10} = \tfrac{21}{2}\sqrt{10}.$$

and the height of the triangle is

$$IS + HO = \tfrac{6}{5}\sqrt{10} + 3\sqrt{10} = \tfrac{21}{5}\sqrt{10}.$$

Hence the area of the triangle FIL is given by

$$\tfrac{1}{2}(\text{base} \times \text{height}) = \tfrac{1}{2}\left(\tfrac{21}{2}\sqrt{10} \times \tfrac{21}{5}\sqrt{10}\right) = \tfrac{1}{2}\left(\tfrac{21^2}{10} \times 10\right) = 220.5.$$

For investigation

23.1 Prove (a) that the triangles GPF, HQG and ISH are similar, and (b) that the triangles KML, JRK and ISJ are similar.

23.2 Show that the side FI of the triangle FIL is perpendicular to the side LI.

24. The numbers x, y, p and q are all integers. x and y are variable and p and q are constant and positive. The four integers are related by the equation $xy = px + qy$.

When y takes its maximum possible value, which expression is equal to $y - x$?

A $pq - 1$ B $(p-1)(q-1)$ C $(p+1)(q-1)$ D $(p-1)(q+1)$ E $(p+1)(q+1)$

SOLUTION **D**

Since

$$xy = px + qy,$$

we have

$$xy - qy = px.$$

Therefore

$$(x - q)y = px.$$

Hence

$$y = \frac{px}{x-q} = p + \frac{pq}{x-q}.$$

Since p and q are constant, y attains its maximum value when x has the value that maximises $\frac{pq}{x-q}$. Since $pq > 0$, this is the value of x for which $x - q$ attains its minimum positive value.

Because x and q are integers the minimum positive value of $x - q$ is 1. This occurs for $x = q + 1$. For this value of x, we have $y = p + pq$, which is an integer.

It follows that when y takes its maximum possible value, $y - x = (p + pq) - (q + 1) = p(q + 1) - (q + 1) = (p - 1)(q + 1)$.

FOR INVESTIGATION

24.1 (a) Find all the solutions of the equation

$$xy = 6x + 5y$$

in which x and y are positive integers.

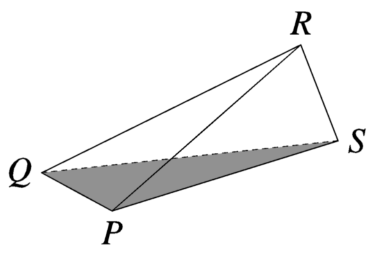
(b) Find the maximum value of y which occurs among these solutions.

(c) Verify that this value of y is $p + pq$ with $p = 6$ and $q = 5$.

25. A drinks carton is formed by arranging four congruent triangles as shown. $QP = RS = 4\,\text{cm}$ and $PR = PS = QR = QS = 10\,\text{cm}$.

What is the volume, in cm^3, of the carton?

A $\frac{16}{3}\sqrt{23}$ B $\frac{4}{3}\sqrt{2}$ C $\frac{128}{25}\sqrt{6}$
D $\frac{13}{2}\sqrt{23}$ E $\frac{8}{3}\sqrt{6}$



Solution **A**

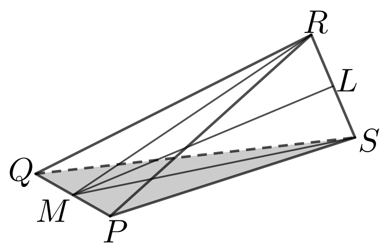
The lengths in this question are given in terms of centimetres. However, for convenience, we will ignore these units in the calculations until we reach the final answer.

The drinks carton is in the shape of a pyramid. We therefore use the fact that the volume of a pyramid is given by the formula

$$\text{volume} = \tfrac{1}{3}(\text{area of base} \times \text{height}).$$

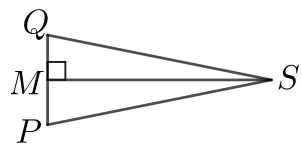
We let M be the midpoint of PQ and let L be the midpoint of RS.

The carton may be thought of as made up of two pyramids, one with the triangle RMS as its base and P as its apex, and the other with RMS as its base and Q as its apex.



Since $PS = QS$, and M is the midpoint of QP, the triangles PMS and QMS are congruent.

It follows that the angles $\angle PMS$ and $\angle QMS$ are equal. Because they are angles on a line, it follows that they are both right angles.



Since $\angle PMS$ is a right angle, PM is the height of the pyramid with base RMS and apex P. Therefore the volume of this pyramid is $\frac{1}{3}(\text{area of } RMS \times PM)$.

Similarly the volume of the pyramid with base RMS and apex Q is $\frac{1}{3}(\text{area of } RMS \times MQ)$.

We let V be the volume of the carton.

We now have that

$$\begin{aligned} V &= \tfrac{1}{3}(\text{area of } RMS \times PM) + \tfrac{1}{3}(\text{area of } RMS \times MQ) \\ &= \tfrac{1}{3}(\text{area of } RMS \times (PM + MQ)) \\ &= \tfrac{1}{3}(\text{area of } RMS \times PQ). \qquad (1) \end{aligned}$$

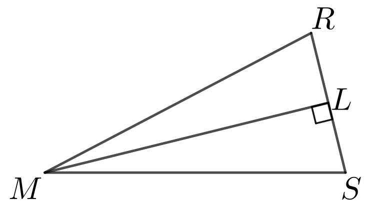
We therefore now calculate the area of the triangle RMS.

Since the triangles PRQ and PSQ are congruent it may be shown that $RM = SM$ (see Problem 25.3).

It follows that the triangles RML and SML are congruent, and hence that ML is at right angles to RS.

Therefore

$$\text{area of } RMS = \tfrac{1}{2}(RS \times ML). \qquad (2)$$



By Pythagoras theorem applied to the right-angled triangle SMP, we have $MS^2 = PS^2 - PM^2 = 10^2 - 2^2 = 100 - 4 = 96$. Therefore, by Pythagoras' Theorem applied to the right-angled triangle SLM, we have $ML^2 = MS^2 - SL^2 = 96 - 2^2 = 92$. Hence $ML = \sqrt{92} = 2\sqrt{23}$.

We can now deduce, by (1) and (2) that

$$\begin{aligned} V &= \tfrac{1}{3}(\tfrac{1}{2}(RS \times ML) \times PQ) \\ &= \tfrac{1}{6}(RS \times ML \times PQ) \\ &= \tfrac{1}{6}(4 \times 2\sqrt{23} \times 4) \\ &= \tfrac{16}{3}\sqrt{23}. \end{aligned}$$

Hence the volume of the carton is $\frac{16}{3}\sqrt{23}\,\text{cm}^3$.

FOR INVESTIGATION

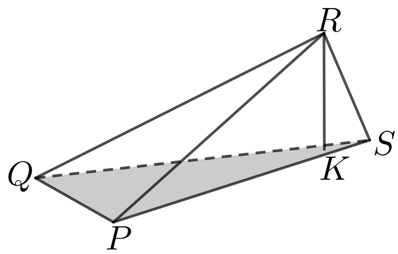
25.1 How can it be proved that the volume of a pyramid is given by the formula

$$\tfrac{1}{3}(\text{area of base} \times \text{height})\,?$$

25.2 Another method for solving this problem is to regard the carton as a pyramid with the triangle PQS as its base and R as its apex. Then the volume of the carton is given

$$\tfrac{1}{3}(\text{area of } PQS \times RK),$$

where K is the foot of the perpendicular from R to the triangle PQS.



Use this method to find the volume of the carton.

25.3 Explain how $RM = SM$ follows from the fact that the triangles PRQ and PSQ are congruent.